

Relativistic Pöschl-Teller and Rosen-Morse problems

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November 11, 2008

Abstract

It is shown that the new family of geometric models of the relativistic oscillator [1], which generalize the anti-de Sitter model, leads to relativistic Pöschl-Teller or Rosen-Morse problems.

One of the simplest (1+1) geometric models is that of the (classical or quantum) relativistic harmonic oscillator (RHO). Based on phenomenological [2] and group theoretical [3, 4] arguments, this has been defined as a free system on the anti-de Sitter static background. There exists a (3+1) anti-de Sitter static metric [2] which reproduces the classical equations of motion of the non-relativistic isotropic harmonic oscillator. This metric can be restricted to a (1+1) metric which gives the classical equation of the one-dimensional non-relativistic harmonic oscillator (NRHO). Moreover, the corresponding quantum model has an equidistant discrete energy spectrum with a ground state energy larger, but approaching $\omega/2$ in the non-relativistic limit (in natural units $\hbar = c = 1$) [5].

In a previous article [1] we have generalized this model to the family of (1+1) models depending on a real parameter λ which has the metrics given by

$$ds^2 = g_{00}dt^2 + g_{11}dx^2 = \frac{1 + (1 + \lambda)\omega^2x^2}{1 + \lambda\omega^2x^2}dt^2 - \frac{1 + (1 + \lambda)\omega^2x^2}{(1 + \lambda\omega^2x^2)^2}dx^2, \quad (1)$$

where ω is the frequency. This parametrization has been defined in order to obtain the exact anti-de Sitter metric for $\lambda = -1$. Therein we have studied the scalar field ϕ of the mass m , defined on a suitable space domain, D , and minimally coupled with the gravitational field [6] given by (1). Since the energy is conserved, the Klein-Gordon equation, $(\square + m^2)\phi = 0$, admits the set of fundamental solutions (of positive and negative frequency)

$$\phi_E^{(+)} = \frac{1}{\sqrt{2E}} e^{-iEt} U_E(x), \quad \phi^{(-)} = (\phi^{(+)})^*, \quad (2)$$

which must be orthogonal with respect to the relativistic scalar product [6]. This reduces to the following scalar product of the wave functions U

$$(U, U') = \int_D dx \mu(x) U^*(x) U'(x), \quad (3)$$

where

$$\mu(x) = \sqrt{-g(x)} g^{00}(x) = \frac{1}{\sqrt{1 + \lambda \omega^2 x^2}}, \quad g = \det(g_{\mu\nu}). \quad (4)$$

By solving the Klein-Gordon equations we have shown that the models with $\lambda > 0$ have mixed energy spectra, with a finite discrete sequence and a continuous part, while for $\lambda \leq 0$ these spectra are countable [1]. However, despite of their different relativistic behavior, all these models have the same non-relativistic limit, namely the NRHO of the frequency ω . For this reason we shall use the name of relativistic oscillators (RO) for all the models with $\lambda \neq -1$, understanding that the RHO is only that of the anti-de Sitter metric.

In general, any (1+1) static background admits a special natural frame in which the metric is a conformal transformation of the Minkowski flat metric. This new frame can be obtained by changing the space coordinate

$$x \rightarrow \hat{x} = \int dx \mu(x) + \text{const.} \quad (5)$$

Then

$$\hat{g}_{00}(\hat{x}) = -\hat{g}_{11}(\hat{x}) = \sqrt{-\hat{g}(\hat{x})}, \quad \hat{\mu}(\hat{x}) = 1 \quad (6)$$

and, therefore, the scalar product (3) becomes the usual one.

Our aim is to show that, in this frame, the Klein-Gordon equation of our RO will involve relativistic symmetric Pöschl-Teller (PT) potentials for $\lambda < 0$ and symmetric Rosen-Morse (RM) potentials for $\lambda > 0$ [7]. When $\lambda = 0$ the relativistic potential will be proportional with that of the NRHO.

Let us first consider the case of $\lambda < 0$ and denote

$$\lambda = -\epsilon^2, \quad \hat{\omega} = \epsilon\omega, \quad \epsilon \geq 0. \quad (7)$$

Then, by changing the space coordinate according to (5) and (4) we obtain

$$\hat{x} = \frac{1}{\hat{\omega}} \arcsin \hat{\omega} x \quad (8)$$

and the new form of the line element

$$ds^2 = \left(1 + \frac{1}{\epsilon^2} \tan^2 \hat{\omega} \hat{x}\right) (dt^2 - d\hat{x}^2). \quad (9)$$

The functions U are defined on $D = (-\pi/2\hat{\omega}, \pi/2\hat{\omega})$ because of the singularities of the metric which determine the event horizon. The Klein-Gordon equation,

$$\left(-\frac{d^2}{d\hat{x}^2} + \frac{m^2}{\epsilon^2} \tan^2 \hat{\omega} \hat{x}\right) U(\hat{x}) = (E^2 - m^2)U(\hat{x}), \quad (10)$$

has a countable discrete energy spectrum [1]. Therefore, the energy eigenfunctions must be square integrable with respect to the usual scalar product in the coordinate \hat{x} . These have the form

$$U_n(\hat{x}) = N_{n_s, s} \cos^k \hat{\omega} \hat{x} \sin^s \hat{\omega} \hat{x} F(-n_s, k + s + n_s, s + \frac{1}{2}, \sin^2 \hat{\omega} \hat{x}), \quad (11)$$

where $N_{n_s, s}$ is the normalization factor, and

$$k = \frac{1}{2} \left[1 + \sqrt{1 + 4 \frac{m^2}{\epsilon^2 \hat{\omega}^2}} \right] > 1, \quad (12)$$

is the positive solution of the equation

$$k(k-1) = \frac{m^2}{\epsilon^2 \hat{\omega}^2}. \quad (13)$$

The quantum numbers $n_s = 0, 1, 2, \dots$ and $s = 0, 1$ can be embedded into the main quantum number $n = 2n_s + s$, which take even values if $s = 0$ and odd values for $s = 1$. The energy levels are given by

$$E_n^2 = m^2 + \hat{\omega}^2 \left[2k \left(n + \frac{1}{2} \right) + n^2 \right], \quad n = 0, 1, 2, \dots \quad (14)$$

if $\epsilon \neq 1$, and by

$$E_n = \hat{\omega}(k + n) \quad (15)$$

in the case of the RHO [5], when $\epsilon = 1$. According to (13), the second term of the left-hand side of (20) can be written as

$$V_{PT}(\hat{x}) = k(k-1)\hat{\omega}^2 \tanh^2 \hat{\omega} \hat{x}. \quad (16)$$

This will be called the relativistic (symmetric) PT potential since the solutions (11) coincide with those given by the Schrödinger equation with the non-relativistic PT potential $V_{PT}/2m$. Hence, for $\epsilon > 0$ our RO are systems of relativistic massive scalar particles confined to wells, as in the non-relativistic case, but having new energy spectra and another parametrization which depends on m , ω and ϵ . We note that our new parameter ϵ allows to choose the desired well width, $\pi/\epsilon\omega$, when the frequency ω is fixed. Therefore, this will be a supplementary fit parameter in the problems of geometric confinement.

For $\lambda > 0$ we change the significance of ϵ and we put

$$\lambda = \epsilon^2, \quad \hat{\omega} = \epsilon\omega, \quad \epsilon \geq 0, \quad (17)$$

so that, according to (5) and (4), the change of the space coordinate will be given by

$$x = \frac{1}{\hat{\omega}} \sinh \hat{\omega} \hat{x}. \quad (18)$$

Now, the line element is

$$ds^2 = \left(1 + \frac{1}{\epsilon^2} \tanh^2 \hat{\omega} \hat{x}\right) (dt^2 - d\hat{x}^2) \quad (19)$$

and $D = (-\infty, \infty)$. The Klein-Gordon equation

$$\left(-\frac{d^2}{d\hat{x}^2} + \frac{m^2}{\epsilon^2} \tanh^2 \hat{\omega} \hat{x}\right) U(\hat{x}) = (E^2 - m^2)U(\hat{x}) \quad (20)$$

has a mixed energy spectrum [1]. The square integrable energy eigenfunctions of the finite discrete spectrum are

$$U_n(\hat{x}) = N_{n_s, s} \cosh^{-k'} \hat{\omega} \hat{x} \sinh^s \hat{\omega} \hat{x} F(-n_s, -k' + s + n_s, s + \frac{1}{2}, -\sinh^2 \hat{\omega} \hat{x}), \quad (21)$$

where now

$$k' = \frac{1}{2} \left[-1 + \sqrt{1 + 4 \frac{m^2}{\epsilon^2 \hat{\omega}^2}} \right] \quad (22)$$

is the positive solution of the equation

$$k'(k' + 1) = \frac{m^2}{\epsilon^2 \hat{\omega}^2}. \quad (23)$$

The quantum number $n = 2n_s + s$ can take the values $n = 0, 1, \dots, n_{max} < k'$. One can verify that the discrete spectrum is included in the domain $[m, m\sqrt{1 + 1/\epsilon^2})$ since the energy levels are given by

$$E_n^2 = m^2 + \hat{\omega}^2 [2k'(n + \frac{1}{2}) - n^2], \quad n = 0, 1, 2, \dots, n_{max} \quad (24)$$

The continuous spectrum is $[m\sqrt{1 + 1/\epsilon^2}, \infty)$ while the corresponding generalized energy eigenfunction are

$$U_\nu(\hat{x}) = N_\nu \cosh^{-k'} \hat{\omega} \hat{x} \sinh^s \hat{\omega} \hat{x} F(-k' + s + i\nu, -k' + s - i\nu, s + \frac{1}{2}, -\sinh^2 \hat{\omega} \hat{x}), \quad (25)$$

where

$$\nu(E) = \frac{1}{2\hat{\omega}} \sqrt{E^2 - m^2 \left(1 + \frac{1}{\epsilon^2}\right)} \in [0, \infty). \quad (26)$$

As in the previous case, we shall say that

$$V_{RM}(\hat{x}) = k'(k' + 1) \hat{\omega}^2 \tanh^2 \hat{\omega} \hat{x} \quad (27)$$

is the relativistic (symmetric) RM potential. The solutions (21) and (25) are the same as those given by the non-relativistic RM potential $V_{RM}/2m$. Of course, the relativistic energy spectra differ from the non-relativistic ones. We note that the number of the discrete energy levels is determined by the values of m/ω (i.e. $mc^2/\hbar\omega$ in usual units) and ϵ defined according to (17). Moreover, for a fixed m , the domains of the discrete and continuous spectra are given by ϵ only.

We have shown [1] that our family of RO is continuous in $\lambda = 0$. This means that the limits for $\epsilon \rightarrow 0$ of the PT and RM systems must coincide. Indeed, then we have $\hat{x} \rightarrow x$ and $k \rightarrow \infty$, $k' \rightarrow \infty$ but

$$\lim_{\epsilon \rightarrow 0} \epsilon^2 k = \lim_{\epsilon \rightarrow 0} \epsilon^2 k' = \frac{m}{\omega}. \quad (28)$$

Therefore, we can verify that, in this limit, we obtain the relativistic potential

$$V(x) = \lim_{\epsilon \rightarrow 0} V_{PT}(\hat{x}) = \lim_{\epsilon \rightarrow 0} V_{RM}(\hat{x}) = m^2 \omega^2 x^2 \quad (29)$$

which gives the same energy eigenfunctions as those of the NRHO and the levels

$$E_n^2 = m^2 + 2m\omega(n + \frac{1}{2}). \quad (30)$$

Obviously, the NRHO potential is $V/2m$.

The conclusion is that our family of metrics (1), which depend on the parameter λ , generates relativistic PT or RM problems, in the special frames (t, \hat{x}) . The PT and RM potentials have similar forms being proportional with $m^2/|\lambda|$. On the other hand, it is interesting that this parameter (or the parameter ϵ related to it) has not a direct non-relativistic equivalent since, in this limit, all the RO become NRHO and, consequently, the terms involving λ disappear [1]. However, its physical significance results from the analysis of the relativistic effects, as we have seen from the previous discussion concerning the behavior of the relativistic PT or RM systems.

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